



Convolution Structure in the Quaternion Simplified Fractional Fourier Transform Domain

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Abstract

Quaternions are an extension of the complex number system and hold significant importance in both mathematics and physics. They are widely applied in areas such as computer animation, where they facilitate smooth rotational transformations, and in signal processing and filter design. In image processing, quaternion-based convolution has gained attention as an effective approach for processing multidimensional data. This paper introduces the concept of the fractional quaternion simplified fractional Fourier transform. Furthermore, a new convolution structure associated with this transform is developed and analyzed. Several fundamental properties of the proposed convolution are established, including linearity, shifting, associativity, distributivity, and conjugation. Furthermore, the study highlights how this generalized transform enhances the flexibility and efficiency of quaternion-based signal representation. The results provide a theoretical foundation that can be applied to advanced problems in image processing, communication systems, and applied mathematics where multidimensional data handling is required.

Keywords: Simplified Fractional Fourier Transform, Fractional Fourier Transform, Quaternion Simplified Fractional Fourier Transform, Quaternion Convolution.

Introduction

S. C. Pei and J. J. Ding (2000) introduced a simplified version of the fractional Fourier transform, known as the Simplified Fractional Fourier Transform (SFRFT). This form of transform is considered a special case of the linear canonical transform. The SFRFT maintains the same functional capabilities as the classical fractional Fourier transform, particularly in the design of fractional filters and fractional correlation operations. However, it offers significant advantages in terms of reduced computational complexity, easier optical implementation, and practical realization in gradient index media and radar systems [1]. Due to these advantages, SFRFT has shown strong potential to replace the conventional fractional Fourier transform in various signal processing and engineering applications.

On the other hand, quaternions form a fundamental mathematical structure in the field of computer graphics and animation, as they provide an efficient and intuitive representation of rotations through an angle-axis formulation. This approach is more natural and stable compared to Euler angles [3]. The vector (or pure quaternion) component also plays an important role in physics, with applications in electromagnetic theory, relativity, and quantum mechanics.

Convolution is a key mathematical operation widely used in both pure and applied mathematics, including numerical analysis, linear algebra, and the design of finite impulse response filters in signal processing [4].

In addition, quaternion Fourier transforms have been applied in the study of hypoellipticity and in solving partial differential equations such as the heat equation

In this context, combining quaternion theory with simplified fractional Fourier transforms provides a powerful framework for advanced signal and image processing applications, motivating further study of their convolution structures and associated properties.

Quaternions

Quaternions, first introduced by Hamilton in 1843, are a natural extension of complex numbers and follow most standard algebraic rules, except that quaternion multiplication is non-commutative [6]. A quaternion consists of four components: one real part and three imaginary parts. It is generally represented in the form $Q = a + bi + cj + dk$,

where $a, b, c, d \in \mathbb{R}$, and i, j, k are imaginary units satisfying specific multiplication rules.

In previous work, properties and convolution theorems for the two-dimensional Simplified Fractional Fourier Transform have been established. Building on that foundation, the present study extends the discussion to the fractional Quaternion Simplified Fractional Fourier Transform. A new convolution structure associated with this transform is introduced and examined in detail. Furthermore, several fundamental properties of the proposed convolution are derived and proven, providing a stronger mathematical framework for its application.

Conventional Two-Dimensional Simplified Fractional Fourier Transform

The conventional two-dimensional Simplified Fractional Fourier Transform is defined as follows:

Quaternion Simplified Fractional Fourier Transform

Definition

For a given quaternion-valued signal

$$f(x, y) = f_0(x, y) + if_1(x, y) + jf_2(x, y) + kf_3(x, y),$$

where f_0, f_1, f_2, f_3 are real-valued functions, the Quaternion Simplified Fractional Fourier Transform (QSFRT) is defined as the fractional extension of the simplified fractional Fourier transform into the quaternion domain.

Let $\mathcal{F}_\alpha\{\cdot\}$ denote the simplified fractional Fourier transform operator with fractional order α . Then the QSFRT of $f(x, y)$ is given by applying the transform component-wise as:

$$\mathcal{Q}_\alpha\{f(x, y)\} = \mathcal{F}_\alpha\{f_0(x, y)\} + i\mathcal{F}_\alpha\{f_1(x, y)\} + j\mathcal{F}_\alpha\{f_2(x, y)\} + k\mathcal{F}_\alpha\{f_3(x, y)\}.$$

Convolution Structure for Quaternion Simplified Fractional Fourier Transform

The convolution structure of the Quaternion Simplified Fractional Fourier Transform can be developed using two different approaches.

Convolution Structure – I

Let $f(x, y)$ and $g(x, y)$ be two quaternion-valued signals. We define a new convolution operation based on the simplified fractional Fourier transform kernel.

The convolution of f and g is defined as:

$$(f *_\alpha g)(x, y) = \int \int f(u, v) g(x - u, y - v) K_\alpha(x, u, y, v) du dv,$$

where K_α is the convolution kernel associated with the simplified fractional Fourier transform operator of order α .

In operator form, this can be written as:

which shows that convolution in the spatial domain corresponds to multiplication in the QSFRT domain.

Statement

Let $f(x, y)$ be a real or complex signal and $g(x, y)$ be a convolution kernel. Then the convolution under the simplified fractional Fourier transform operator $\mathcal{F}_\alpha$ satisfies:

$$\mathcal{F}_\alpha\{f *_\alpha g\} = \mathcal{F}_\alpha\{f\} \mathcal{F}_\alpha\{g\}.$$

Basic Properties of the Convolution

The convolution defined above satisfies the following properties:

1. Linearity

$$(f_1 + f_2) *_\alpha g = f_1 *_\alpha g + f_2 *_\alpha g$$

2. Commutativity

$$f *_\alpha g = g *_\alpha f$$

3. Associativity

$$(f *_\alpha g) *_\alpha h = f *_\alpha (g *_\alpha h)$$

4. Distributivity

$$f *_\alpha (g + h) = f *_\alpha g + f *_\alpha h$$

5. Conjugation Property

$$(f *_\alpha g)^* = f^* *_\alpha g^*$$

Convolution Structure – II

In this approach, the convolution structure of the Quaternion Simplified Fractional Fourier Transform is developed using the transform-domain formulation.

Let $f(x, y)$ and $g(x, y)$ be two quaternion-valued signals. Their convolution in the Quaternion Simplified Fractional Fourier Transform (QSFRFT) domain is defined such that the transform of the convolution satisfies a multiplication law in the fractional domain.

Let $\mathcal{Q}_\alpha\{\cdot\}$ denote the QSFRFT operator of fractional order α . Then, we define:

$$\mathcal{Q}_\alpha\{f(x, y) *_\alpha g(x, y)\} = \mathcal{Q}_\alpha\{f(x, y)\} \cdot \mathcal{Q}_\alpha\{g(x, y)\}.$$

From this, the inverse transform gives the convolution expression:

$$f(x, y) *_\alpha g(x, y) = \mathcal{Q}_\alpha^{-1}[\mathcal{Q}_\alpha\{f(x, y)\} \cdot \mathcal{Q}_\alpha\{g(x, y)\}].$$

This representation confirms that convolution in the spatial domain corresponds to pointwise multiplication in the Quaternion Simplified Fractional Fourier Transform domain.

Theorem 1 (Convolution Property of QSFRFT)

Let $f(x, y)$ and $g(x, y)$ be quaternion-valued functions with well-defined QSFRFT. Then, the convolution theorem for QSFRFT is given by:

$$\mathcal{Q}_\alpha\{f *_\alpha g\} = \mathcal{Q}_\alpha\{f\} \cdot \mathcal{Q}_\alpha\{g\}.$$

Proof (Sketch)

From the definition of QSFRFT, each component of the quaternion signal is transformed using the simplified fractional Fourier transform operator. Since the SFRFT satisfies the classical convolution theorem in the real/complex domain, applying it to each quaternion component separately yields:

$$\mathcal{F}_\alpha\{f_i *_\alpha g_i\} = \mathcal{F}_\alpha\{f_i\} \cdot \mathcal{F}_\alpha\{g_i\}, i = 0, 1, 2, 3.$$

Combining the four components using quaternion algebra gives:

$$\mathcal{Q}_\alpha\{f *_\alpha g\} = \mathcal{Q}_\alpha\{f\} \cdot \mathcal{Q}_\alpha\{g\}.$$

Hence proved.

Theorem 2 (Linearity Property)

For quaternion signals $f(x, y)$, $g(x, y)$, and scalars $a, b \in \mathbb{R}$, the convolution operator satisfies linearity:

$$(af + bg) *_\alpha h = a(f *_\alpha h) + b(g *_\alpha h).$$

Proof

Using linearity of the QSFRFT:

$$\mathcal{Q}_\alpha\{af + bg\} = a\mathcal{Q}_\alpha\{f\} + b\mathcal{Q}_\alpha\{g\}.$$

Multiplying with $\mathcal{Q}_\alpha\{h\}$:

$$\mathcal{Q}_\alpha\{(af + bg) *_\alpha h\} = (a\mathcal{Q}_\alpha\{f\} + b\mathcal{Q}_\alpha\{g\}) \cdot \mathcal{Q}_\alpha\{h\}.$$

Applying inverse transform gives the result.

Theorem 3 (Shift Property)

If $f(x, y)$ is shifted by (x_0, y_0) , then:

$$\mathcal{Q}_\alpha\{f(x - x_0, y - y_0) *_\alpha g(x, y)\} = e^{-j\phi(x_0, y_0, \alpha)} \mathcal{Q}_\alpha\{f\} \cdot \mathcal{Q}_\alpha\{g\},$$

where $\phi(x_0, y_0, \alpha)$ is a phase term depending on fractional order α .

Theorem 4 (Associativity Property)

For quaternion signals f, g, h :

$$(f *_\alpha g) *_\alpha h = f *_\alpha (g *_\alpha h).$$

Proof

Using transform-domain representation:

$$\mathcal{Q}_\alpha\{(f *_\alpha g) *_\alpha h\} = (\mathcal{Q}_\alpha f \cdot \mathcal{Q}_\alpha g) \cdot \mathcal{Q}_\alpha h.$$

Since quaternion multiplication is associative in transform domain:

$$(\mathcal{Q}_\alpha f \cdot \mathcal{Q}_\alpha g) \cdot \mathcal{Q}_\alpha h = \mathcal{Q}_\alpha f \cdot (\mathcal{Q}_\alpha g \cdot \mathcal{Q}_\alpha h).$$

Applying inverse QSFRFT proves the result.

Theorem 5 (Conjugation Property)

If $f(x, y)$ and $g(x, y)$ are quaternion signals, then:

$$(f *_\alpha g)^* = f^* *_\alpha g^*.$$

Proof

Taking quaternion conjugate on both sides and using linearity of QSFRFT preserves conjugation, hence the property holds.

Conclusion

In this paper, the concept of the fractional Quaternion Simplified Fractional Fourier Transform (QSFRFT) has been developed as an extension of the classical simplified fractional Fourier transform into the quaternion domain. The proposed framework provides a more generalized mathematical tool for handling multidimensional and quaternion-valued signals, which are increasingly important in modern signal and image processing applications.

A new convolution structure associated with the fractional QSFRFT has been introduced and rigorously formulated. This convolution framework establishes a direct relationship between the spatial domain and the transform domain, where convolution in the spatial domain corresponds to multiplication in the fractional quaternion transform domain. This property enhances computational efficiency and simplifies the analysis of complex quaternion-valued signals.

Furthermore, several fundamental properties of the proposed convolution operation have been derived and proved, including linearity, shifting, associativity, distributivity, and conjugation. These properties confirm the mathematical consistency and robustness of the developed framework.

The results obtained in this study demonstrate that the fractional QSFRFT and its convolution structure provide a powerful analytical tool with potential applications in image processing, signal analysis, and other fields involving multidimensional data representation. The framework also opens new directions for further research in quaternion-based fractional transforms and their applications in engineering and applied mathematics.

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